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A boundary integral approach to analyze the viscous scattering of a pressure wave by a rigid body

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Abstract

The paper provides boundary integral equations for solving the problem of viscous scattering of a pressure wave by a rigid body. By using this mathematical tool uniqueness and existence theorems are proved. Since the boundary conditions are written in terms of velocities, vector boundary integral equations are obtained for solving the problem. The paper introduces single-layer viscous potentials and also a stress tensor. Correspondingly, a viscous double-layer potential is defined. The properties of all these potentials are investigated.

By representing the scattered field as a combination of a single-layer viscous potential and a double-layer viscous potential the problem is reduced to the solution of a singular vectorial integral equation of Fredholm type of the second kind.

In the case where the stress vector on the boundary is the main quantity of interest the corresponding boundary singular integral equation is proved to have a unique solution.

Keywords: Acoustical scattering; Viscous fluid; Integral equation; MEMS

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0. Introduction

The first of the two articles dedicated by Kress to acoustic scattering in [1] starts with the sentence: "Mathematical acoustics is concerned with the modeling of sound waves considered as small perturbations in fluid (or gas)". For classical mathematical acoustics a more precise assertion has to be made by changing the word "fluid" to "inviscid fluid". Thus, in most papers dedicated to acoustic scattering, the basic equation is the scalar Helmholtz's equation satisfied by pressure in the case of time harmonic acoustic waves in inviscid (nonviscous) fluids. This approach resulted in a lot of theoretical and practical results included in classical books by Morse and Ingard [2] and Pierce [3]. Theoretical classical and modern results connected to direct and inverse scattering problems are presented in the excellent book by Colton and Kress [4]; an update and a reproduction in more compact form of the main facts contained in this book are included in the chapters authored by Kress in [1]. The linearity of the basic equation and the infinity of the domain, involved in many applications, made the Boundary Integral Method the most suitable mathematical instrument for approaching classical acoustical problems and, correspondingly, the Boundary Element Method as a powerful tool in computational acoustic analysis. There are a lot of books covering all the aspects of the problem including detailed discussions of the basic theory, numerical algorithms and practical engineering applications [5] and [6].

The technology of micro-electro-mechanical systems (MEMS) has raised the problem of study of the motion of gases at microscale geometries (microfluidics). Thus, when the dimensions of the body are of the order of the boundary layer thickness (as is the case of microphones built in MEMS technology) the viscous effects cannot be neglected. The same is the case for underwater acoustic waves which we expect to be strongly influenced by the water viscosity. These are sufficient reasons to justify the development of a viscous acoustic scattering theory based on the linearization of the equations for the motion of viscous fluids.

The inclusion of viscous effects in acoustics is a subject not very often approached. The book by Pierce [3] contains a chapter discussing the dissipative processes devoted especially to explain attenuation of sound waves. In Ref. [7] a solution is given to the problem of diffraction of a plane sound wave by a grating in the case of a viscous compressible fluid having important applications in MEMS. Finally, Ref. [8] gives a solution to the problem of the viscous scattering of a pressure wave by an artificial hair-like biomimetic acoustic velocity sensor

allowing the calculation of the fluid tractions on the sensor.

The important difference in mathematically approaching viscous acoustics as compared with the classical nonviscous case is associated with boundary conditions: in the classical case the single condition on a hard (rigid) body is the canceling of the normal derivative of the total pressure on the body's surface, describing in fact the nonpenetration of the air particles through the solid boundary; in the viscous case, the commonly used condition is the nonslip condition asserting that the fluid particles are sticking on the body's surface. Consequently, instead of working with the pressure as the main unknown function we have to use the velocity components. As a result instead of a scalar function we have to determine a vectorial field.

[Section 1](#) starts with the derivation of the basic equation satisfied by the velocity field in the case of the linearized form of Navier–Stokes equations for compressible isentropic flow in the case of a harmonic dependence in time. This way the pressure is eliminated from the problem by means of the continuity equation. Once the velocity field is determined, the pressure can be obtained directly from the continuity equation. Afterwards, the viscous scattering problem is formulated and a uniqueness theorem is proved. The scattering problem is in fact an exterior Dirichlet-type problem for the system of partial differential equations equivalent to the basic equation of viscous acoustics.

[Section 2](#) is dedicated to studying the properties of the viscous acoustic single- and double-layer potentials. The approach is classical: reciprocal relationships of Lorentz type, fundamental solution and the Green's formulas for bounded and unbounded domains. Viscous acoustic potentials of single and double-layer are defined and their properties are presented. By using a representation of velocities in terms of the single-layer potential and a double-layer potential a singular (vectorial) integral equation of Fredholm type of the second kind on the boundary of the body is obtained. In [Section 3](#) the singular integral equation satisfied by traction (surface force) on the boundary is studied. By using some results from the theory of 2D singular integral equations it is proved that the corresponding singular integral equation has a unique solution.

There results that the viscous acoustic scattering problem is well-posed. Its solution can be obtained by solving the singular (vectorial) integral equation obtained in the case where the physical stress is considered. Having proved that the integral equation is uniquely solvable its solutions can be obtained by using some specific numerical methods.

In order to facilitate the reading of the paper the most important results of the theory of 2D singular integral equations, used in paper, are added in an Appendix.

1. Viscous effect in acoustic waves

1.1. The equations of the motion of a compressible viscous fluid

The motion of a viscous fluid is described by the continuity equation

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' \mathbf{V}') = 0, \quad (1)$$

and the momentum equation

$$\rho' \left(\frac{\partial}{\partial t} + \mathbf{V}' \cdot \nabla \right) \mathbf{V}' - \nabla \cdot \boldsymbol{\Sigma}' = \mathbf{0} \quad (2)$$

and “a state” equation. Here \mathbf{V}' denotes velocity, ρ' is density and the stress operator $\boldsymbol{\Sigma}'$ has the components

(3)

$$\Sigma'_{ij} \equiv \sigma_{ij}[P', \mathbf{V}'] := \left[-P' + \left(\mu_B - \frac{2}{3}\mu \right) \nabla \cdot \mathbf{V}' \right] \delta_{ij} + \mu \left(\frac{\partial V'_i}{\partial x_j} + \frac{\partial V'_j}{\partial x_i} \right).$$

Also, μ and μ_B are the shear and bulk viscosities [3] and P' is the pressure.

In the case of isentropic flow the density is a function of pressure alone such that the state equation can be expressed as

$$\rho' = \rho'(P'), \quad (4)$$

where

$$\frac{d\rho'}{dP'} = \frac{1}{c'^2}, \quad (5)$$

c' being the isentropic speed of sound. By using formulas (4) and (5) the continuity equation can be written as

$$\frac{1}{c'^2} \frac{\partial P'}{\partial t} + \nabla \cdot (\rho' \mathbf{V}') = 0. \quad (6)$$

For a viscous fluid we have usually the nonslip boundary condition

$$\mathbf{V}'(\mathbf{x}, t) = \mathbf{0} \quad (7)$$

on any immobile solid surface. It is to be noticed that in some cases (as in the case of a slightly rarefied compressible gas) the first-order slip velocity conditions at solid boundaries are considered instead of nonslip conditions(7).

1.2. The equations of the motion of a viscous fluid in linear acoustic approximation

Acoustic disturbances are usually regarded as small-amplitude perturbations to an ambient state [2] and [3]. For the fluid the ambient state is characterized by the constant values $(P_0, \rho_0, \mathbf{V}_0)$. In the case where the coordinate system is chosen so that the unperturbed fluid is at rest $\mathbf{V}_0 = \mathbf{0}$ and the dependent variables can be written as

$$\begin{aligned} \mathbf{V}' &= \mathbf{V}'', & |\mathbf{V}''|^2 &\ll |\mathbf{V}''|, \\ P' &= P_0 + P'', & |P''| &\ll |P_0|, \\ \rho' &= \rho_0 + \rho'', & |\rho''| &\ll \rho_0, \end{aligned}$$

where \mathbf{V}'' , P'' and ρ'' represent the perturbations of velocity, pressure and density. The acoustic approximation of the flow equations are obtained by neglecting the product of perturbations. Eq. (6) then becomes

$$\frac{1}{c_0^2} \frac{\partial P''}{\partial t} + \nabla \cdot \mathbf{V}'' = 0, \quad (8)$$

where $c_0 = \sqrt{P_0/\rho_0}$ is the unperturbed isentropic speed of sound. Also, Eq. (2) can be written as

$$\rho_0 \frac{\partial \mathbf{V}''}{\partial t} - \nabla \cdot \Sigma'' = \mathbf{0}. \quad (9)$$

Consider now the case where all the physical variables are harmonic in time with the same angular frequency $\omega = 2\pi f$. The case of general time dependence can be obtained, after analyzing each frequency separately, by Fourier superposition. In the case of simple harmonic oscillations in time the pressure and velocity perturbations can be written as

$$\{P''(\mathbf{x}, t), \mathbf{V}''(\mathbf{x}, t)\} = \{P(\mathbf{x}), \mathbf{V}(\mathbf{x})\} \exp(-i\omega t).$$

Eqs. (8), (9) become

$$\frac{\partial V_j}{\partial x_j} = \frac{i\omega P}{c_0^2 \rho_0}, \quad (10)$$

$$-i\omega \rho_0 V_i = \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i = 1, 2, 3, \quad (11)$$

where P and \mathbf{V} denote the amplitudes of the pressure and velocity perturbations and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(P, \mathbf{V})$. The physical stress operator $\boldsymbol{\sigma}$, taking into account the relationship (10), can be written as

$$\sigma_{ij}[\mathbf{V}] = \lambda \left(\frac{\partial V_k}{\partial x_k} \right) \delta_{ij} + \mu e_{ij}, \quad e_{ij} = \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right), \quad (12)$$

where

$$\lambda \equiv \lambda_1 + i\lambda_2 = \left(\mu_B - \frac{2}{3}\mu \right) + i \frac{\rho_0 c_0^2}{\omega}.$$

Also, the kinematic viscosities have been denoted by ν and ν' :

$$\nu = \frac{\mu}{\rho_0}, \quad \nu' = \frac{\mu_B}{\rho_0} + \frac{4\mu}{3\rho_0}.$$

The system of Eqs. (10), (11) yields the following equation for the pressure:

$$[\Delta + k^2]P = 0, \quad (13)$$

where

$$k = \frac{\omega}{c_0 \sqrt{1 - i\omega\nu'/c_0^2}}, \quad \text{Im}(k) \geq 0,$$

and the velocity field's equation

$$\mu \Delta \mathbf{V} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{V} + i\rho_0 \omega \mathbf{V} = 0. \quad (14)$$

Since $\nabla \cdot \mathbf{V} = i\omega/(\rho_0 c_0^2)P$ by applying to Eq. (14) the operator $[\Delta + k^2]$ there results *the basic equation*

$$[\Delta + k^2][\Delta + k^{*2}]\mathbf{V} = 0, \quad (15)$$

where

$$k^* = \sqrt{\frac{i\omega}{\nu}}, \quad \text{Im}(k^*) \geq 0.$$

Despite the compact form of the basic equation we prefer to use the system of Eqs. (10), (11) and (12) which contains all the physical variables $\mathbf{V}, P, \boldsymbol{\sigma}$.

1.3. The direct scattering problem

The viscous scattering of time-harmonic acoustic waves by a rigid bounded obstacle $D \subset \mathbb{R}^3$ is modeled by the following problem: Given an incident field $(P^{in}, \mathbf{V}^{in})$ satisfying the system (10), (11) in some domain containing \overline{D} , find the scattered field (P^s, \mathbf{V}^s) as a radiating solution to the system (10), (11) in $\mathbb{R}^3 \setminus D$ such that the total velocity field

$$\mathbf{V} = \mathbf{V}^{in} + \mathbf{V}^s$$

satisfies the boundary condition $\mathbf{V} = 0$ on ∂D .

Here ∂D denotes the boundary surface of the domain D , assumed of class C^2 .

1.3.1. The incident field

The incident field can be an incoming plane pressure wave

$$P^{in}(\mathbf{x}) = \rho_0 c_0^2 \exp\{ik\mathbf{m} \cdot \mathbf{x}\}, \quad (16)$$

where \mathbf{m} is the unit vector of the propagation direction of the wave. It can be verified directly that the function $P^{in}(\mathbf{x})$ satisfies the pressure equation (13). Also, Eqs. (10) and (11) provide the corresponding velocity field as

$$\mathbf{V}^{in}(\mathbf{x}) = i\delta k c_0^2 \mathbf{m} \exp\{ik\mathbf{m} \cdot \mathbf{x}\}, \quad (17)$$

where it has been denoted by

$$\delta = \frac{1 + (v - v')i\omega/c_0^2}{i\omega - vk^2}.$$

In other problems the incident fields can characterize a point source in the domain $\mathbb{R}^3 \setminus D$.

Clearly, the direct scattering problem is in fact a special *exterior Dirichlet problem* for the bounded domain D :

Find a vectorial field $\mathbf{v}^s(\mathbf{x}) \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$, solution of the basic equation

$$\mu\Delta\mathbf{v} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{v} + i\rho_0\omega\mathbf{v} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \quad (18)$$

that satisfies the boundary condition

$$\mathbf{v} = \mathbf{f} \quad \text{on } \partial D,$$

(\mathbf{f} being a continuous vectorial field defined on ∂D) and vanishing at infinity at least like $1/|\mathbf{x}|$.

1.3.2. The uniqueness theorem

Theorem 1

In the case we have

$$v_j = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \frac{\partial v_j}{\partial x_k} = o\left(\frac{1}{|\mathbf{x}|}\right), \quad j, k = 1, 2, 3 \quad (19)$$

the exterior Dirichlet problem for Eq. (18) has at most one solution.

We must show that the homogeneous boundary condition $\mathbf{v}=\mathbf{0}$ on ∂D implies that \mathbf{v} vanishes identically. Denote $\Omega_r := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < r\}$ and $D_r = (\mathbb{R}^3 \setminus D) \cap \Omega_r$. Then

$$-i\omega\rho_0 \int_{D_r} v_i \bar{v}_i \, dx = \int_{D_r} \bar{v}_i \frac{\partial \sigma_{ij}}{\partial x_j} \, dx.$$

But

$$\bar{v}_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} (\bar{v}_i \sigma_{ij}) - \frac{\partial \bar{v}_i}{\partial x_j} \sigma_{ij}.$$

Hence

$$-i\omega\rho_0 \int_{D_r} v_i \bar{v}_i \, dx = \int_{\partial D} \bar{v}_i t_i \, ds + \int_{\partial \Omega_r} \bar{v}_i t_i \, ds - \int_{D_r} \sigma_{ij} \bar{e}_{ij} \, dx,$$

whereby $t_i = \sigma_{ij} n_j$ was denoted by the traction on the surface, and by \mathbf{n} the outer normal unit vector. Substituting σ_{ij} by its expression given by Eq. (12) there results

$$\begin{aligned} & \int_{D_r} \{\lambda_1 (\nabla \cdot \mathbf{v}) (\overline{\nabla \cdot \mathbf{v}}) \, dx + 2\mu e_{ij} \bar{e}_{ij} - i\omega\rho_0 v_i \bar{v}_i\} \, dx \\ & = \int_{\partial D} \bar{v}_i t_i \, ds + \int_{\partial \Omega_r} \bar{v}_i t_i \, ds. \end{aligned}$$

By virtue of (19) the integral over $\partial\Omega_r$ tends to zero when $r \rightarrow \infty$. Therefore, the limit of the right-hand side and hence the limit of the left-hand side exist and are equal. Taking the real part in the resulting relationship we obtain for the homogeneous problem,

$$2\mu \int_{\mathbb{R}^3 \setminus D} \sum_{ij} |e_{ij}|^2 dx + \left(\mu_B - \frac{2}{3} \mu \right) \int_{\mathbb{R}^3 \setminus D} |\nabla \cdot \mathbf{v}|^2 dx = 0. \quad (20)$$

Now, by using the inequality $|z_1+z_2+z_3|^2 \leq 3(|z_1|^2+|z_2|^2+|z_3|^2)$ there results

$$\frac{2}{3} \mu |\nabla \cdot \mathbf{v}|^2 \leq 2\mu \left(\left| \frac{\partial v_1}{\partial x_1} \right|^2 + \left| \frac{\partial v_2}{\partial x_2} \right|^2 + \left| \frac{\partial v_3}{\partial x_3} \right|^2 \right).$$

Finally, formula (20) becomes

$$\int_{\mathbb{R}^3 \setminus D} \left\{ 2\mu \sum_{i \neq j} |e_{ij}|^2 dx + \mu_B \sum_{i=1}^3 \left| \frac{\partial v_i}{\partial x_i} \right|^2 \right\} dx \leq 0.$$

Hence $|e_{ij}| = |\partial v_i / \partial x_j| = 0$. Taking into consideration the boundary conditions there results $\mathbf{v} = \mathbf{0}$ in $\mathbb{R}^3 \setminus D$.

2. Green's formula and viscous acoustic layer potentials

2.1. The reciprocal identity

Since the incident field (P^i, \mathbf{V}^i) satisfies the basic equations system (10)+(11) the scattered field (ρ^s, \mathbf{v}^s) will be the solution of the system

$$\frac{\partial v_j^s}{\partial x_j} = \frac{i\omega p^s}{c_0^2 \rho_0} \quad (21)$$

$$-i\omega \rho_0 v_i^s = \frac{\partial \sigma_{ij}^s}{\partial x_j}, \quad i = 1, 2, 3, \quad (22)$$

where

$$\sigma_{ij}^s = \sigma_{ij}[\mathbf{v}^s] := \lambda \left(\frac{\partial v_k^s}{\partial x_k} \right) \delta_{ij} + \mu \left(\frac{\partial v_i^s}{\partial x_j} + \frac{\partial v_j^s}{\partial x_i} \right) \quad (23)$$

is the stress operator.

Let us assume that (\mathbf{u}, p) and (\mathbf{u}', p') are two solutions of Eqs. (21), (22) and (23) with the associated stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$, respectively, and compute

$$\begin{aligned} u'_i \frac{\partial \sigma_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_j} (u'_i \sigma_{ij}) - \sigma_{ij} \frac{\partial u'_i}{\partial x_j} \\ &= \frac{\partial}{\partial x_j} (u'_i \sigma_{ij}) - (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{u}') \\ &\quad - \mu \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \frac{\partial u'_i}{\partial x_j} \right). \end{aligned} \quad (24)$$

Interchanging the roles of \mathbf{u} and \mathbf{u}' there results

$$(25)$$

$$\begin{aligned}
u_i \frac{\partial \sigma'_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_j} (u_i \sigma'_{ij}) - \sigma'_{ij} \frac{\partial u_i}{\partial x_j} \\
&= \frac{\partial}{\partial x_j} (u_i \sigma'_{ij}) - (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{u}') \\
&\quad - \mu \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} \right).
\end{aligned}$$

Subtracting (25) from (24) we find

$$\frac{\partial}{\partial x_j} (u'_i \sigma_{ij} - u_i \sigma'_{ij}) = u'_i \frac{\partial \sigma_{ij}}{\partial x_j} - u_i \frac{\partial \sigma'_{ij}}{\partial x_j} = 0.$$

This yields the reciprocal identity

$$\nabla \cdot (\mathbf{u}' \cdot \boldsymbol{\sigma} - \mathbf{u} \cdot \boldsymbol{\sigma}') = 0 \tag{26}$$

known as Lorentz' formula in fluid mechanics [10] and Betti's formula in linear elasticity [11].

Another form of reciprocal identity can be obtained by integrating (26) over a fluid domain D bounded by the closed surface ∂D and then using the divergence theorem

$$\int_{\partial D} \mathbf{u}' \cdot \mathbf{f} \, ds = \int_{\partial D} \mathbf{u} \cdot \mathbf{f}' \, ds,$$

where $\mathbf{f} = \boldsymbol{\sigma} \cdot \mathbf{n}$ and $\mathbf{f}' = \boldsymbol{\sigma}' \cdot \mathbf{n}$ are the surface forces (tractions) exerted on ∂D , and \mathbf{n} is the unit normal vector pointing outside D .

2.2. The fundamental solution

To determine the fundamental solution of Eq. (18) we consider the equation

$$\mu \Delta \mathbf{v} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{v} + i \rho_0 \omega \mathbf{v} = -\mathbf{b} \delta(\mathbf{x}) \quad \text{in } \mathbb{R}^3 \setminus \{0\}, \tag{27}$$

\mathbf{b} being a constant vector and $\delta(\mathbf{x})$ the Dirac's function.

By taking the Fourier Transform of Eq. (27) there results

$$\mu \alpha^2 \tilde{\mathbf{v}} + (\lambda + \mu) \boldsymbol{\alpha} (\boldsymbol{\alpha} \cdot \tilde{\mathbf{v}}) - i \omega \rho_0 \tilde{\mathbf{v}} = \mathbf{b}. \tag{28}$$

Here $\alpha = |\boldsymbol{\alpha}|$ and

$$\tilde{\mathbf{v}}(\boldsymbol{\alpha}) = \mathcal{F}[\mathbf{v}] := \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{x}) \exp(-i \boldsymbol{\alpha} \cdot \mathbf{x}) \, dx.$$

Eq. (28) yields

$$\boldsymbol{\alpha} \cdot \tilde{\mathbf{v}} = \frac{\boldsymbol{\alpha} \cdot \mathbf{b}}{(\lambda + 2\mu)\alpha^2 - i \omega \rho_0}$$

and

$$\begin{aligned}
\tilde{\mathbf{v}}(\boldsymbol{\alpha}) &= \frac{1}{\mu \alpha^2 - i \omega \rho_0 / \mu} \frac{\mathbf{b}}{\mu(\lambda + 2\mu)} \\
&\quad \times \frac{\boldsymbol{\alpha} (\boldsymbol{\alpha} \cdot \mathbf{b})}{[\alpha^2 - i \omega \rho_0 / \mu][\alpha^2 - i \omega \rho_0 / (\lambda + 2\mu)]}.
\end{aligned}$$

By using the formulas

$$\mathcal{F}^{-1} \left[\frac{1}{\alpha^2 - k^{*2}} \right] = \frac{\exp(ik^* |\mathbf{x}|)}{4\pi |\mathbf{x}|},$$

$$\mathcal{F}^{-1} \left[\frac{1}{(\alpha^2 - k^2)(\alpha^2 - k^{*2})} \right] = \frac{\exp(ik|\mathbf{x}|) - \exp(ik^*|\mathbf{x}|)}{4\pi(k^2 - k^{*2})|\mathbf{x}|},$$

there results

$$v_i = S_{ij} b_j, \quad (29)$$

where

$$S_{ij}(\mathbf{x}) = \frac{\exp(ik^*|\mathbf{x}|)}{4\pi\mu|\mathbf{x}|} \delta_{ij} + \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \frac{\partial^2}{\partial x_i \partial x_j} \times \frac{\exp(ik|\mathbf{x}|) - \exp(ik^*|\mathbf{x}|)}{4\pi(k^2 - k^{*2})|\mathbf{x}|}. \quad (30)$$

2.3. Stress tensor corresponding to the fundamental solution

Substituting the fundamental solution (29) into the expression of the stress (12) there results

$$\sigma_{ij}[v] = T_{ijq} b_q, \quad (31)$$

where

$$T_{ijq}(\mathbf{x}) = \left(\frac{\lambda}{\lambda + 2\mu} \delta_{ij} \frac{\partial}{\partial x_q} + \delta_{iq} \frac{\partial}{\partial x_j} + \delta_{jq} \frac{\partial}{\partial x_i} \right) \frac{\exp(ik^*|\mathbf{x}|)}{4\pi|\mathbf{x}|} - \delta_{ij} \frac{\lambda(\lambda + \mu)}{\mu(\lambda + 2\mu)} k^2 \frac{\partial}{\partial x_q} \frac{\exp(ik|\mathbf{x}|) - \exp(ik^*|\mathbf{x}|)}{4\pi(k^2 - k^{*2})|\mathbf{x}|} + \frac{2(\lambda + \mu)}{(\lambda + 2\mu)} \frac{\partial^3}{\partial x_i \partial x_j \partial x_q} \frac{\exp(ik|\mathbf{x}|) - \exp(ik^*|\mathbf{x}|)}{4\pi(k^2 - k^{*2})|\mathbf{x}|}. \quad (32)$$

2.4. The Green's formula (bounded domain)

The Lorentz reciprocal formula (26) yields

$$\frac{\partial}{\partial x_q} (u'_i \sigma_{iq} - u_i \sigma'_{iq}) = 0.$$

Identifying

$$u'_i(\mathbf{x}) = S_{ij}(\mathbf{x}_0, \mathbf{x}) b_j,$$

$$\sigma'_{iq}(\mathbf{x}) = T_{ijq}(\mathbf{x}_0, \mathbf{x}) b_j,$$

where

$$S_{ij}(\mathbf{x}_0, \mathbf{x}) = S_{ij}(\mathbf{x} - \mathbf{x}_0),$$

$$T_{ijq}(\mathbf{x}_0, \mathbf{x}) = T_{ijq}(\mathbf{x} - \mathbf{x}_0),$$

and discarding the arbitrary constant \mathbf{b} there results

$$\frac{\partial}{\partial x_q} [S_{ij}(\mathbf{x}_0, \mathbf{x}) \sigma_{iq}(\mathbf{x}) - u_i(\mathbf{x}) T_{ijq}(\mathbf{x}_0, \mathbf{x})] = 0. \quad (33)$$

The relationship (33) is integrated over the domain D . In the case the point \mathbf{x}_0 is selected outside $D \cup \partial D$ the function within the square bracket in (33) is regular in D and using the divergence theorem there results

$$\int_{\partial D} [S_{ij}(\mathbf{x}_0, \mathbf{x})\sigma_{iq}(\mathbf{x}) - u_i(\mathbf{x})T_{ijq}(\mathbf{x}_0, \mathbf{x})]n_q \, ds = 0, \quad (34)$$

where $n_q = n_q(\mathbf{x})$ and $ds = dS(\mathbf{x})$. In (34) as well as in all subsequent equations, the unit normal vector \mathbf{n} is directed outside the domain D .

In the case where the point \mathbf{x}_0 is selected in the interior of the domain D and Σ_ε denotes the surface of the sphere of radius ε centered at \mathbf{x}_0 , the function within the square bracket in (33) is regular through the domain bounded by the surfaces ∂D and Σ_ε and there results

$$\int_{\partial D \cup \Sigma_\varepsilon} [S_{ij}(\mathbf{x}_0, \mathbf{x})\sigma_{iq}(\mathbf{x}) - u_i(\mathbf{x})T_{ijq}(\mathbf{x}_0, \mathbf{x})]n_q \, ds = 0$$

which can also be written as

$$\begin{aligned} & \int_{\partial D} [S_{ij}(\mathbf{x}_0, \mathbf{x})\sigma_{iq}(\mathbf{x}) - u_i(\mathbf{x})T_{ijq}(\mathbf{x}_0, \mathbf{x})]n_q \, ds \\ & = \int_{\Sigma_\varepsilon} [-S_{ij}(\mathbf{x}_0, \mathbf{x})\sigma_{iq}(\mathbf{x}) + u_i(\mathbf{x})T_{ijq}(\mathbf{x}_0, \mathbf{x})]n_q \, ds. \end{aligned} \quad (35)$$

In the last integral in this equation the change of variables $\mathbf{x} = \mathbf{x}_0 + \varepsilon\hat{\mathbf{x}}$ ($\mathbf{n} = -\hat{\mathbf{x}}$, $dS(\mathbf{x}) = \varepsilon^2 dS(\hat{\mathbf{x}})$) will be performed. As $\varepsilon \rightarrow 0$, the values of \mathbf{u} and $\boldsymbol{\sigma}$ over Σ_ε tend to their corresponding values at the center of the sphere, $\mathbf{u}(\mathbf{x}_0)$ and $\boldsymbol{\sigma}(\mathbf{x}_0)$, respectively. Therefore,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} [-S_{ij}(\mathbf{x}_0, \mathbf{x})\sigma_{iq}(\mathbf{x}) + u_i(\mathbf{x})T_{ijq}(\mathbf{x}_0, \mathbf{x})]n_q \, dS(\mathbf{x}) \\ & = \sigma_{iq}(\mathbf{x}_0) \lim_{\varepsilon \rightarrow 0} \int_{\Sigma_1} S_{ij}(\hat{\mathbf{x}})\hat{x}_q \varepsilon^2 \, dS(\hat{\mathbf{x}}) \\ & \quad - u_i(\mathbf{x}_0) \lim_{\varepsilon \rightarrow 0} \int_{\Sigma_1} T_{ijq}(\hat{\mathbf{x}})\hat{x}_q \varepsilon^2 \, dS(\hat{\mathbf{x}}). \end{aligned}$$

Substituting the expressions of $S_{ij}(\hat{\mathbf{x}})$ and $T_{ijq}(\hat{\mathbf{x}})$ and using the relationship

$$\int_{\Sigma_1} \hat{x}_i \hat{x}_q \, dS(\hat{\mathbf{x}}) = \begin{cases} 0, & i \neq q, \\ 4\pi/3, & i = q, \end{cases}$$

there results

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} [-S_{ij}(\mathbf{x} - \mathbf{x}_0)\sigma_{iq}(\mathbf{x}) \\ & \quad + u_i(\mathbf{x})T_{ijq}(\mathbf{x} - \mathbf{x}_0)]n_q \, dS(\mathbf{x}) = u_j(\mathbf{x}_0). \end{aligned}$$

Finally, the relationship (35) becomes

$$\begin{aligned} u_j(\mathbf{x}_0) & = \int_{\partial D} S_{ij}(\mathbf{x}_0, \mathbf{x})t_i(\mathbf{x}) \, ds \\ & \quad - \int_{\partial D} u_i(\mathbf{x})K_{ji}(\mathbf{x}_0, \mathbf{x}, \mathbf{n}) \, ds, \end{aligned}$$

where $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ is the traction (surface force) on ∂D and

(36)

$$\begin{aligned}
K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') &= \left(\frac{\lambda}{\lambda + 2\mu} n'_j \frac{\partial}{\partial x'_i} + \delta_{ij} \frac{\partial}{\partial n'} + n'_i \frac{\partial}{\partial x'_j} \right) \\
&\times \frac{\exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi|\mathbf{x} - \mathbf{x}'|} - \frac{\lambda(\lambda + \mu)}{\mu(\lambda + 2\mu)} k^2 n'_j \frac{\partial}{\partial x'_i} \\
&\times \frac{\exp(ik|\mathbf{x} - \mathbf{x}'|) - \exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi(k^2 - k^{*2})|\mathbf{x} - \mathbf{x}'|} \\
&+ \frac{2(\lambda + \mu)}{\lambda + 2\mu} \frac{\partial}{\partial n'} \frac{\partial^2}{\partial x'_i \partial x'_j} \\
&\times \frac{\exp(ik|\mathbf{x} - \mathbf{x}'|) - \exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi(k^2 - k^{*2})|\mathbf{x} - \mathbf{x}'|}.
\end{aligned}$$

There results

$$\left. \begin{aligned}
u_i(\mathbf{x}), & \quad \mathbf{x} \in D \\
u_i(\mathbf{x})/2, & \quad \mathbf{x} \in \partial D \\
0, & \quad \mathbf{x} \in \mathbb{R}^3 \setminus \bar{D}
\end{aligned} \right\} = \int_{\partial D} S_{ij}(\mathbf{x}, \mathbf{x}') t_j(\mathbf{x}') ds' \tag{37}$$

$$- \int_{\partial D} K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') u_j(\mathbf{x}') ds'.$$

Here we have denoted $\mathbf{n}' = \mathbf{n}(\mathbf{x}')$ and $ds' = dS(\mathbf{x}')$. Eq. (37) provides a representation of a flow in terms of boundary distributions involving the tensors \mathbf{S} and \mathbf{K} .

Remark 1

In the case where the point $\mathbf{x} \in \partial D$ the second integral in formula (37) is not convergent (as an improper integral); since it results by taking the limit when the radius of the small excluded sphere $\epsilon \rightarrow 0$ it has to be considered in the sense of a principal value (see Appendix A).

2.5. The Green's formula (unbounded domain)

In the case where the flow domain coincides with $\mathbb{R}^3 \setminus \bar{D}$ we write

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^{in}(\mathbf{x}) + \mathbf{v}(\mathbf{x}),$$

where

$$\lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{v}(\mathbf{x}) = 0.$$

Also denote by Σ_R^0 the sphere of radius R centered at the origin. By applying formula (37) to the function $\mathbf{v}(\mathbf{x})$ and the domain delimited by the surfaces ∂D and Σ_R^0 there results

$$\begin{aligned}
v_i(\mathbf{x}) &= - \int_{\partial D} [S_{ij}(\mathbf{x}, \mathbf{x}') t_j(\mathbf{x}') - K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') v_j(\mathbf{x}')] ds' \\
&+ \int_{\Sigma_R^0} [S_{ij}(\mathbf{x}, \mathbf{x}') t_j(\mathbf{x}') - K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') v_j(\mathbf{x}')] ds',
\end{aligned}$$

the unit normal vector \mathbf{n}' being directed outside domain D . But

$$\lim_{R \rightarrow \infty} \int_{\Sigma_R^0} [S_{ij}(\mathbf{x}, \mathbf{x}') t_j(\mathbf{x}') - K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') v_j(\mathbf{x}')] ds' = 0.$$

Finally, the Green's formula for the infinite domain $\mathbb{R}^3 \setminus \bar{D}$ has the form

$$\left. \begin{array}{ll} V_i(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3 \setminus \overline{D} \\ V_i(\mathbf{x})/2, & \mathbf{x} \in \partial D \\ 0, & \mathbf{x} \in D \end{array} \right\} = V_i^m - \int_{\partial D} S_{ij}(\mathbf{x}, \mathbf{x}') t_j(\mathbf{x}') ds' + \int_{\partial D} K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') V_j(\mathbf{x}') ds'. \quad (38)$$

2.6. Viscous acoustic layer potentials

The general representation formulas (37) and (38) for the solution of the basic equation (18) suggests the introduction of viscous acoustic layer potentials. These potentials will be defined in close analogy with the acoustic layer potentials [1] and [9].

2.6.1. Viscous acoustic single-layer potential

Given an integrable vector field Φ the vectorial integral $\mathbf{u}(\mathbf{x})$ of components

$$u_i(\mathbf{x}) := \int_{\partial D} S_{ij}(\mathbf{x}, \mathbf{x}') \phi_j(\mathbf{x}') ds', \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial D \quad (39)$$

is called a *viscous acoustic single-layer potential* of density Φ . It is a solution of the basic equation (18) in D and $\mathbb{R}^3 \setminus \overline{D}$ and is vanishing at infinity. Physically, the viscous acoustic single-layer potential gives the velocity field corresponding to a stress force (traction) along the surface ∂D given by the function $\Phi(\mathbf{x}')$.

By using the formulas

$$\exp(ik^*|\mathbf{x}|) = 1 + ik^*|\mathbf{x}| + O(|\mathbf{x}|^2), \quad (40)$$

$$\frac{\exp(ik|\mathbf{x}|) - \exp(ik^*|\mathbf{x}|)}{4\pi(k^2 - k^{*2})|\mathbf{x}|} = \frac{i}{k + k^*} - \frac{|\mathbf{x}|}{2} + O(|\mathbf{x}|^2), \quad (41)$$

we can write

$$S_{ij}(\mathbf{x}, \mathbf{x}') = S_{ij}^0(\mathbf{x}, \mathbf{x}') + S_{ij}^R(\mathbf{x}, \mathbf{x}'),$$

where the singular part can be written as

$$S_{ij}^0(\mathbf{x}, \mathbf{x}') = \frac{1}{8\pi(\lambda + 2\mu)|\mathbf{x} - \mathbf{x}'|} \times \left[(\lambda + 3\mu)\delta_{ij} + (\lambda + \mu) \frac{(x_i - x'_i)(x_j - x'_j)}{|\mathbf{x} - \mathbf{x}'|^2} \right]$$

and the regular part is

$$S_{ij}^R(\mathbf{x}, \mathbf{x}') = \frac{\exp(ik^*|\mathbf{x} - \mathbf{x}'|) - 1}{4\pi|\mathbf{x} - \mathbf{x}'|} \delta_{ij} + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\partial^2}{\partial x_i \partial x_j} \times \left[\frac{\exp(ik|\mathbf{x} - \mathbf{x}'|) - \exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi(k^2 - k^{*2})|\mathbf{x} - \mathbf{x}'|} + \frac{|\mathbf{x} - \mathbf{x}'|}{2} \right].$$

The singular part $S_{ij}^0(\mathbf{x})$ has only integrable singularities of the same type as the simple-layer harmonic potential. Therefore, the regularity and jump relationships for the solutions for the viscous acoustic single-layer potential will be similar to those corresponding to the harmonic single-layer potential. For example, in the case $\Phi \in C^0(\partial D)$, the viscous acoustic single-layer is a continuous vector in the whole space.

Let \mathbf{x} be a point in space, and a small area element with \mathbf{n} as the direction of the normal. Then, the stress vector acting on this area element corresponding to the displacement field given by a single-layer potential can be written in the form

$$t_i[\mathbf{u}, \mathbf{n}](\mathbf{x}) \equiv n_j \sigma_{ij}[\mathbf{u}] = \int_{\partial D} K_{ij}^*(\mathbf{x}, \mathbf{x}', \mathbf{n}) \varphi_j(\mathbf{x}') ds',$$

where

$$\begin{aligned} K_{ij}^*(\mathbf{x}, \mathbf{x}', \mathbf{n}) = & \left(\delta_{ij} \frac{\partial}{\partial n} + n_j \frac{\partial}{\partial x_i} + \frac{\lambda}{\lambda + 2\mu} n_i \frac{\partial}{\partial x_j} \right) \\ & \times \frac{\exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi|\mathbf{x} - \mathbf{x}'|} + \frac{2(\lambda + \mu)}{\lambda + 2\mu} \frac{\partial}{\partial n} \frac{\partial^2}{\partial x_i \partial x_j} \\ & \times \frac{\exp(ik|\mathbf{x} - \mathbf{x}'|) - \exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi(k^2 - k^{*2})|\mathbf{x} - \mathbf{x}'|} \\ & - \frac{\lambda(\lambda + \mu)}{\lambda + 2\mu} k^2 n_i \frac{\partial}{\partial x_j} \\ & \times \frac{\exp(ik|\mathbf{x} - \mathbf{x}'|) - \exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi(k^2 - k^{*2})|\mathbf{x} - \mathbf{x}'|}. \end{aligned} \quad (42)$$

2.6.2. Viscous acoustic double-layer potential

The *viscous acoustic double-layer potential*, of integrable density Φ , is the vectorial field \mathbf{v} having the components

$$v_i = \int_{\partial D} K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') \varphi_j(\mathbf{x}') ds', \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial D. \quad (43)$$

Physically, the double-layer potential represents the velocity field in the entire space, produced by the concentrated moment $\Phi(\mathbf{x}')$ on the surface ∂D with normal \mathbf{n}' . The vectorial field \mathbf{v} is a solution of the basic equation (18) in D and $\mathbb{R}^3 \setminus \bar{D}$ and is also vanishing at infinity.

Using again the formulas (40) and (41) we can separate the singular and regular components of the kernel

$$K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') = K_{ij}^0(\mathbf{x}, \mathbf{x}', \mathbf{n}') + K_{ij}^R(\mathbf{x}, \mathbf{x}', \mathbf{n}'),$$

where

$$\begin{aligned} K_{ij}^0(\mathbf{x}, \mathbf{x}', \mathbf{n}') = & \frac{\mathbf{n}' \cdot (\mathbf{x} - \mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|^3} \left[\left(\frac{\lambda + \mu}{\lambda + 2\mu} - 1 \right) \delta_{ij} \right. \\ & \left. - \frac{3(\lambda + \mu)}{\lambda + 2\mu} \frac{(x_i - x'_i)(x_j - x'_j)}{|\mathbf{x} - \mathbf{x}'|^2} \right] \\ & + \frac{2\mu}{\lambda + 2\mu} \frac{n'_j(x_i - x'_i) - n'_i(x_j - x'_j)}{8\pi|\mathbf{x} - \mathbf{x}'|^3}, \end{aligned} \quad (44)$$

$$\begin{aligned} K_{ij}^R(\mathbf{x}, \mathbf{x}', \mathbf{n}') = & \left(\delta_{ij} \frac{\partial}{\partial n'} + n'_i \frac{\partial}{\partial x'_j} + \frac{\lambda}{\lambda + 2\mu} n'_j \frac{\partial}{\partial x'_i} \right) \\ & \times \frac{\exp(ik^*|\mathbf{x} - \mathbf{x}'|) - 1}{4\pi|\mathbf{x} - \mathbf{x}'|} + \frac{2(\lambda + \mu)}{(\lambda + 2\mu)} \frac{\partial}{\partial n'} \frac{\partial^2}{\partial x'_i \partial x'_j} \\ & \times \left[\frac{\exp(ik|\mathbf{x} - \mathbf{x}'|) - \exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi(k^2 - k^{*2})|\mathbf{x} - \mathbf{x}'|} + \frac{|\mathbf{x} - \mathbf{x}'|}{2} \right] \\ & - \frac{\lambda(\lambda + \mu)}{\mu(\lambda + 2\mu)} k^2 n'_j \frac{\partial}{\partial x'_i} \\ & \times \frac{\exp(ik|\mathbf{x} - \mathbf{x}'|) - \exp(ik^*|\mathbf{x} - \mathbf{x}'|)}{4\pi(k^2 - k^{*2})|\mathbf{x} - \mathbf{x}'|}. \end{aligned} \quad (45)$$

In the case $\partial D \in C^2$ we have

$$|\mathbf{n}' \cdot (\mathbf{x}' - \mathbf{x})| \leq c |\mathbf{x}' - \mathbf{x}|^2$$

for all $\mathbf{x}', \mathbf{x} \in \partial D$ and some positive constant c depending on ∂D (see, for example, [9, Theorem 2.2]).

Consequently, the first term in (44) yields an integral operator with weakly singular kernel. The second term, in general case, has a nonintegrable singularity for $\mathbf{x} = \mathbf{x}' \in \partial D$ and, correspondingly, the direct value of the double-layer potential (appearing in the middle case in formulas (37) and (38)) can be understood only as a principal value integral.

Now, the general representation formula (38) can be used for determining the limit values of the generalized double-layer potential on the point $\mathbf{x}_0 \in \partial D$.

$$\begin{aligned} F_i(\mathbf{x}_0) + \int_{\partial D} K_{ij}(\mathbf{x}_0^+, \mathbf{x}', \mathbf{n}') V_j(\mathbf{x}') ds &= V_i(\mathbf{x}_0^+), \quad \mathbf{x}_0^+ \in \mathbb{R}^3 \setminus \bar{D}, \\ F_i(\mathbf{x}_0) + \int_{\partial D} K_{ij}(\mathbf{x}_0, \mathbf{x}', \mathbf{n}') V_j(\mathbf{x}') ds &= V_i(\mathbf{x}_0^+)/2, \quad \mathbf{x}_0 \in \partial D, \\ F_i(\mathbf{x}_0) + \int_{\partial D} K_{ij}(\mathbf{x}_0^-, \mathbf{x}', \mathbf{n}') V_j(\mathbf{x}') ds &= 0, \quad \mathbf{x}_0^- \in D, \end{aligned} \quad (46)$$

where we denoted

$$F_i(\mathbf{x}_0) := V_i^{(in)}(\mathbf{x}_0) - \int_{\partial D} S_{ij}(\mathbf{x}_0, \mathbf{x}') t_j(\mathbf{x}') ds'.$$

Here the continuity of the incoming velocity field and of the single-layer potential across the surface ∂D has been used. By eliminating $F_i(\mathbf{x}_0)$ between formulas (46) there results the limit values of the generalized double-layer potential along the surface ∂D :

$$\mathbf{v}_{\pm}(\mathbf{x}_0) = \mathbf{v}_0(\mathbf{x}_0) \pm \frac{1}{2} \mathbf{V}(\mathbf{x}_0), \quad \mathbf{x}_0 \in \partial D, \quad (47)$$

where by $\mathbf{v}_0(\mathbf{x}_0)$ the vector of components

$$v_{0i}(\mathbf{x}_0) = \int_{\partial D} K_{ij}(\mathbf{x}_0, \mathbf{x}', \mathbf{n}') V_j(\mathbf{x}') ds$$

has been denoted. In the general case the integrals in this formula have to be considered as 2D principal value integrals.

2.6.3. Limiting values of the stress operator for a single-layer potential

The following relationship between the kernel of the double-layer potential (36) and the kernel of the integral relationship giving the generalized stress operator applied to the single-layer potential (42) proves true:

$$K_{ij}^*(\mathbf{x}, \mathbf{x}', \mathbf{n}) = K_{ji}(\mathbf{x}', \mathbf{x}, \mathbf{n}).$$

Since matrix $\|K_{ij}^*(\mathbf{x}, \mathbf{x}', \mathbf{n})\|$ is obtained from matrix $\|K_{ji}(\mathbf{x}', \mathbf{x}, \mathbf{n})\|$ by interchanging the points \mathbf{x} and \mathbf{x}' and by transposing the elements it will be called the conjugate to $\|K_{ij}\|$.

Let now $\mathbf{x}_0 \in \partial D$, \mathbf{n}_0 the normal unit vector (pointing outside domain D) at the point \mathbf{x}_0 , and the points $\mathbf{x}_0^+ \in \mathbb{R}^3 \setminus \bar{D}$ and $\mathbf{x}_0^- \in D$ on the normal \mathbf{n}_0 in close proximity of the surface ∂D . We can consider the stress vectors generated by the single-layer potential in the points \mathbf{x}_0^{\pm} and the direction \mathbf{n}_0 . Then

$$t_i^+[\mathbf{u}, \mathbf{n}_0](\mathbf{x}_0) = \lim_{\mathbf{x}_0^+ \rightarrow \mathbf{x}_0} \int_{\partial D} K_{ij}^*(\mathbf{x}_0^+, \mathbf{x}', \mathbf{n}_0) \varphi_j(\mathbf{x}') ds',$$

$$t_i^-[\mathbf{u}, \mathbf{n}_0](\mathbf{x}_0) = \lim_{\mathbf{x}_0^- \rightarrow \mathbf{x}_0} \int_{\partial D} K_{ij}^*(\mathbf{x}_0^-, \mathbf{x}', \mathbf{n}_0) \varphi_j(\mathbf{x}') ds'.$$

Due to the above noted conjugation the integral

$$t_i[\mathbf{u}, \mathbf{n}_0](\mathbf{x}_0) = \int_{\partial D} K_{ij}^*(\mathbf{x}_0, \mathbf{x}', \mathbf{n}_0) \varphi_j(\mathbf{x}') \, ds'$$

can be considered only as a principal value.

We can write

$$\begin{aligned} t_{+i}[\mathbf{u}, \mathbf{n}_0](\mathbf{x}_0) &= \lim_{\mathbf{x}_0^+ \rightarrow \mathbf{x}_0} \int_{\partial D} [K_{ij}^*(\mathbf{x}_0^+, \mathbf{x}', \mathbf{n}_0) \\ &\quad + K_{ij}(\mathbf{x}_0^+, \mathbf{x}', \mathbf{n}')] \varphi_j(\mathbf{x}') \, ds' \\ &\quad - \lim_{\mathbf{x}_0^+ \rightarrow \mathbf{x}_0} \int_{\partial D} K_{ij}(\mathbf{x}_0^+, \mathbf{x}', \mathbf{n}') \varphi_j(\mathbf{x}') \, ds'. \end{aligned} \quad (48)$$

By introducing a local system of coordinates and using some estimates similar to those for a harmonic potential it can be shown that the first integral in (48) is continuous for $\mathbf{x}_0^+ \rightarrow \mathbf{x}_0$. The second integral is a double-layer potential. Finally we obtain the formulas

$$\mathbf{t}_{\pm}[\mathbf{u}, \mathbf{n}_0](\mathbf{x}_0) = \mathbf{t}[\mathbf{u}, \mathbf{n}_0](\mathbf{x}_0) \mp \frac{1}{2} \boldsymbol{\varphi}(\mathbf{x}_0), \quad \mathbf{x}_0 \in \partial D.$$

2.6.4. Limiting values of the stress operator for a double-layer potential

The limiting values of the stress operator of a double-layer potential $\mathbf{t}^+[\mathbf{K}, \mathbf{n}_0](\mathbf{x}_0)$ and $\mathbf{t}^-[\mathbf{K}, \mathbf{n}_0](\mathbf{x}_0)$ are connected by the Lyapunov–Tauber theorem: *if the limiting value of the stress operator for a double-layer potential exists on one side of the surface ∂D , the limiting value exists on the other side as well and these limiting values coincide.* A proof for this theorem can be found in [12].

2.6.5. A summary of the jump relationships

We have

$$\mathbf{u}_+ = \mathbf{u}_-, \quad \mathbf{t}_+[\mathbf{u}] - \mathbf{t}_-[\mathbf{u}] = -\boldsymbol{\varphi} \quad \text{on } \partial D \quad (49)$$

for the viscous acoustic single-layer potential, and

$$\mathbf{v}_+ - \mathbf{v}_- = \boldsymbol{\varphi}, \quad \mathbf{t}_+[\mathbf{v}] = \mathbf{t}_-[\mathbf{v}] \quad \text{on } \partial D \quad (50)$$

for viscous acoustic double-layer potential.

2.6.6. Single- and double-layer operators on ∂D

We define the viscous acoustic single-layer operator $(\mathbf{S}\boldsymbol{\varphi})(\mathbf{x})$ by its components

$$(\mathbf{S}\boldsymbol{\varphi})_i(\mathbf{x}) := \frac{1}{2} \int_{\partial D} S_{ij}(\mathbf{x}, \mathbf{x}') \varphi_j(\mathbf{x}') \, ds', \quad \mathbf{x} \in \partial D.$$

The corresponding generalized traction operator $(\mathbf{K}^*\boldsymbol{\varphi})(\mathbf{x})$ has the components

$$(\mathbf{K}^*\boldsymbol{\varphi})_i(\mathbf{x}) := \frac{1}{2} \int_{\partial D} S_{ij}^*(\mathbf{x}, \mathbf{x}') \varphi_j(\mathbf{x}') \, ds', \quad \mathbf{x} \in \partial D.$$

Similarly, the generalized viscous acoustic double-layer operator $(\mathbf{K}\boldsymbol{\varphi})(\mathbf{x})$ has the components

$$(\mathbf{K}\boldsymbol{\varphi})_i(\mathbf{x}) := \frac{1}{2} \int_{\partial D} K_{ij}(\mathbf{x}, \mathbf{x}', \mathbf{n}') \varphi_j(\mathbf{x}') \, ds', \quad \mathbf{x} \in \partial D.$$

In terms of these operators the jump relations (49) and (50) yield in the case of continuous densities $\boldsymbol{\varphi}$

$$\mathbf{u}_{\pm} = \frac{1}{2} \mathbf{S}\boldsymbol{\varphi}, \quad (\mathbf{t}\mathbf{u})_{\pm} = \frac{1}{2} \mathbf{K}^*\boldsymbol{\varphi} \mp \frac{1}{2} \boldsymbol{\varphi}, \quad \mathbf{v}_{\pm} = \frac{1}{2} \mathbf{K}\boldsymbol{\varphi} \pm \frac{1}{2} \boldsymbol{\varphi}. \quad (51)$$

By interchanging the order of integration there results that the operator \mathbf{S} is self-adjoint and the operator \mathbf{K}^* is the

adjoint of \mathbf{K} with respect to the L^2 vector bilinear form on $C(\partial D)$.

3. Determination of the traction on the body in the direct viscous scattering problem

In many problems involving the interaction of waves with obstacles the main physical element of interest is the traction (stress vector) on the body. In this section a method for determining directly the traction on the body by means of a 2D singular integral equation will be presented. It is to be noted that the viscous acoustic double-layer operator $(\mathbf{K}\Phi)(\mathbf{x})$ is a singular operator.

The representation of the velocity field as a combination of a viscous acoustic double-layer potential and a simple-layer potential, by means of the coupling parameter $\eta > 0$

$$v_i(\mathbf{x}) := \int_{\partial D} \{K_{ij}(\mathbf{x}, \mathbf{x}') + \eta S_{ij}(\mathbf{x}, \mathbf{x}')\} \varphi_j(\mathbf{x}') ds', \quad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\partial D} \quad (52)$$

yields the singular vectorial integral equation

$$\Phi + \mathbf{K}\Phi + \eta \mathbf{S}\Phi = 2\mathbf{f}. \quad (53)$$

In fact (53) is a system of singular integral equations. The system can be written as

$$\begin{aligned} \varphi_i + \int_{\partial D} \varphi_j(\mathbf{x}') \frac{n'_i(x'_j - x_j) - n'_j(x'_i - x_i)}{2\pi r^3} ds' \\ + \int K_{ij}^{reg}(\mathbf{x}, \mathbf{x}') \varphi_j(\mathbf{x}') ds' = 2f_i, \end{aligned} \quad (54)$$

where the regular terms of K_{ij} and ηS_{ij} were included in K_{ij}^{reg} .

3.1. Extension of the Fredholm alternatives to the system of singular integral equations (54)

At the point $\mathbf{x} \in \partial D$ we introduce a local system of coordinates with the ξ_1 and ξ_2 axes taken in the tangent plane and the ξ_3 axis in the direction of an external normal. The unknown vector $\Phi(\mathbf{x})$ is also projected on this system and its component denoted by (Φ_1, Φ_2, Φ_3) . The system (54) becomes

$$\varphi_1(\xi) - a \int_{\partial D} \frac{\xi'_1 - \xi_1}{r^3} \varphi_3(\xi') ds_{\xi'} + T_1(\xi) = 2f_1,$$

$$\varphi_2(\xi) - a \int_{\partial D} \frac{\xi'_2 - \xi_2}{r^3} \varphi_3(\xi') ds_{\xi'} + T_2(\xi) = 2f_2,$$

$$\begin{aligned} \varphi_3(\xi) - a \int_{\partial D} \frac{(\xi'_1 - \xi_1)\varphi_1(\xi') + (\xi'_2 - \xi_2)\varphi_2(\xi')}{r^3} ds_{\xi'} \\ + T_3(\xi) = 2f_3. \end{aligned}$$

Here $T_i(\xi)$ are regular operators. By writing $\xi'_1 - \xi_1 = r \cos \theta$, $\xi'_2 - \xi_2 = r \sin \theta$ the symbol matrix is

$$\Phi = \begin{bmatrix} 1 & 0 & ia \cos \theta \\ 0 & 1 & ia \sin \theta \\ -ia \cos \theta & -ia \sin \theta & 1 \end{bmatrix},$$

where $a = \mu/[2(\lambda + 2\mu)]$. By calculating the determinants $\delta_1, \delta_2, \delta_3$ defined in Appendix A there results

$$\delta_1 = 1, \quad \delta_2 = 1, \quad |\delta_3| = |1 - a^2| > 1 - |a|^2 > \frac{55}{64}.$$

Since all the moduli of minors are bounded from below by $\frac{55}{64}$ the system of singular integral equations has an equivalent regularizing operator and the given singular system can be turned into a regular one. Hence, the

Fredholm theorem does apply to the system (54).

3.2. The existence of the solution of Eq. (53)

Consider the homogeneous equation

$$\phi + \mathbf{K}\phi + \eta \mathbf{S}\phi = 0. \quad (55)$$

In the case we use the representation (52) of the solution, the corresponding velocity field satisfies Eq. (18) in $\mathbb{R}^3 \setminus \partial D$ and also the conditions

$$-\mathbf{v}_- = \boldsymbol{\varphi}, \quad -\mathbf{t}_-[\mathbf{v}] = -\eta \boldsymbol{\varphi} \quad \text{on } \partial D. \quad (56)$$

Then, we can write

$$\frac{\partial}{\partial x_j} (\bar{v}_i \sigma_{ij}[\mathbf{v}]) = \bar{v}_i \frac{\partial \sigma_{ij}[\mathbf{v}]}{\partial x_j} + \frac{\partial \bar{v}_i}{\partial x_j} \sigma_{ij}[\mathbf{v}]. \quad (57)$$

But according to Eq. (22) the first term in the right-hand side of Eq. (57) becomes

$$\bar{v}_i \frac{\partial \sigma_{ij}[\mathbf{v}]}{\partial x_j} = -i\omega \rho_0 |\mathbf{v}|^2, \quad (58)$$

and the second term can be written as

$$\frac{\partial \bar{v}_i}{\partial x_j} \sigma_{ij}[\mathbf{v}] = \lambda |\operatorname{div} \mathbf{v}|^2 + 2\mu \left| \frac{\partial v_i}{\partial x_i} \right|^2 + \frac{\mu}{2} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2. \quad (59)$$

By using the relationships (57), (58) and (59) there results

$$\begin{aligned} \frac{\partial}{\partial x_j} (\bar{v}_i \sigma_{ij}[\mathbf{v}]) &= -i\omega \rho_0 |\mathbf{v}|^2 + \lambda |\operatorname{div} \mathbf{v}|^2 + 2\mu \left| \frac{\partial v_i}{\partial x_i} \right|^2 \\ &\quad + \frac{\mu}{2} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2. \end{aligned}$$

The integral over domain D gives

$$\begin{aligned} \int_{\partial D} \bar{v}_i t_i^- ds &= \int_D \left[-i\omega \rho_0 |\mathbf{v}|^2 + \lambda |\operatorname{div} \mathbf{v}|^2 + 2\mu \left| \frac{\partial v_i}{\partial x_i} \right|^2 \right. \\ &\quad \left. + \frac{\mu}{2} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2 \right] dx. \end{aligned}$$

Taking the real part and using also the conditions (56) there results

$$\begin{aligned} \int_{\partial D} \eta |\boldsymbol{\varphi}|^2 ds + \int_D \left[\left(\mu_B - \frac{2}{3} \mu \right) |\operatorname{div} \mathbf{v}|^2 + 2\mu \left| \frac{\partial v_i}{\partial x_i} \right|^2 \right. \\ \left. + \frac{\mu}{2} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2 \right] dx = 0. \end{aligned}$$

But

$$\int_D \left[\frac{2}{3} \mu |\operatorname{div} \mathbf{v}|^2 - 2\mu \left| \frac{\partial v_i}{\partial x_i} \right|^2 \right] dx \leq 0.$$

By adding the last two relationships there results

$$\int_{\partial D} \eta |\boldsymbol{\varphi}|^2 ds + \int_D \left(\mu_B |\operatorname{div} \mathbf{v}|^2 + \frac{\mu}{2} \sum_{i \neq j} \left| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right|^2 \right) dx \leq 0.$$

Hence, $\boldsymbol{\varphi} = \mathbf{0}$. Since the homogeneous equation (55) has only the trivial solution and the Fredholm's Alternative does apply, it follows that the equation (53) has a unique solution for each continuous function \mathbf{f} .

3.3. The integral equation for the traction on the solid body

The method we used in the last section for obtaining the solution of the direct viscous scattering problem can be considered as an indirect method since it furnishes firstly the density ψ , the velocity field being obtained afterwards by means of the representation formula.

A direct boundary integral approach of the viscous scattering problem is based on the Green's representation formula (37) which gives

$$\left. \begin{array}{l} V_i(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{D} \\ 0, \quad \mathbf{x} \in D \end{array} \right\} = V_i^{in}(\mathbf{x}) - \int_{\partial D} S_{ij}(\mathbf{x}, \mathbf{x}') t_j(\mathbf{x}') ds'. \quad (60)$$

Here we have used the boundary condition $\mathbf{V} = \mathbf{0}$ on the surface ∂D . This way the solution is expressed by means of a viscous acoustic single-layer potential with the density equal to the physical traction on the body's surface \mathbf{t} :

$$\mathbf{S}[\mathbf{t}] = 2\mathbf{V}^{in}(\mathbf{x}). \quad (61)$$

On the other hand, by applying the traction operator to the relationship (60) in domain D we obtain by using the second relationship (51)

$$\mathbf{t} + \mathbf{K}^*[\mathbf{t}] = 2\mathbf{t}[\mathbf{V}^{in}] \quad (62)$$

for the physical traction on surface ∂D . Linearly combining Eqs. (61) and (62) there results

$$\mathbf{t} + \mathbf{K}^*[\mathbf{t}] + \eta \mathbf{S}[\mathbf{t}] = 2\mathbf{t}[\mathbf{V}^{in}] + 2\eta \mathbf{V}^{in}. \quad (63)$$

Thus, we have obtained a singular integral equation of the second kind for determining the traction on the surface ∂D . It is to be noticed that Eq. (63) is the adjoint of the singular integral equation (53). We have proved that Eq. (53) is uniquely solvable. There results that Eq. (63) is also uniquely solvable. The advantage of Eq. (63) is that it directly furnishes the traction (stress vector) on the boundary surface ∂D .

4. Conclusion

This paper presents a theoretical framework for the acoustic scattering problem in a viscous fluid. The tractions (surface force) over the body's surface can be determined directly by solving a vectorial singular integral equation which is uniquely solvable.

The paper can set up the foundation for future numerical implementation.

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Appendix A. 2D principal value integrals and singular integral equations

A.1. Definition of 2D principal-value integrals

Let Ω denote a bounded domain in the plane $x_3=0$, $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$, $r=|\mathbf{x}-\mathbf{x}'|$, $x'_1 - x_1 = r \cos \theta$, $x'_2 - x_2 = r \sin \theta$. Also denote $\Omega_\varepsilon = \Omega \setminus (r < \varepsilon)$. We consider singular integrals of the form

$$\int_{\Omega} \frac{f(\mathbf{x}, \theta)}{r^2} u(\mathbf{x}') d\mathbf{x}' = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \frac{f(\mathbf{x}, \theta)}{r^2} u(\mathbf{x}') d\mathbf{x}'$$

supposing that the limit does exist. Such an integral is called a *principal value integral*, the bounded function $f(\mathbf{x}, \theta)$ is *the characteristic of the singular integral* and $u(\mathbf{x}')$ is a *density function* supposedly *Hölderian*. It should be noted that a principal value integral differs from an improper integral in that the form of the excluded domain is not arbitrary. The ratio

$$k^S(\mathbf{x}, \mathbf{x}') = \frac{f(\mathbf{x}, \theta)}{r^2} \tag{64}$$

is the kernel of the singular integral.

In order to obtain the conditions for the existence of the principal value integral we write

$$\begin{aligned} \int_{\Omega} k^S(\mathbf{x}, \mathbf{x}') u(\mathbf{x}') d\mathbf{x}' &= \int_{\Omega_a} k^S(\mathbf{x}, \mathbf{x}') u(\mathbf{x}') d\mathbf{x}' \\ &+ \int_{r < a} k^S(\mathbf{x}, \mathbf{x}') [u(\mathbf{x}') - u(\mathbf{x})] d\mathbf{x}' \\ &+ u(\mathbf{x}) \int_{r < a} k^S(\mathbf{x}, \mathbf{x}') d\mathbf{x}', \end{aligned}$$

a being an arbitrary positive number. The first integrals are absolutely convergent and the last one can be written as

$$\begin{aligned} \int_{r < a} k^S(\mathbf{x}, \mathbf{x}') d\mathbf{x}' &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < r < a} \frac{f(\mathbf{x}, \theta)}{r} dr d\theta \\ &= \left(\lim_{\varepsilon \rightarrow 0} \ln \frac{a}{\varepsilon} \right) \int_0^{2\pi} f(\mathbf{x}, \theta) d\theta. \end{aligned}$$

Thus, the limit of the last term exists if and only if

$$\int_0^{2\pi} f(\mathbf{x}, \theta) d\theta = 0, \quad (65)$$

and the formula for calculating the principal part integral becomes

$$\begin{aligned} \int_{\Omega} k^S(\mathbf{x}, \mathbf{x}') u(\mathbf{x}') d\mathbf{x}' &= \int_{\Omega_a} k^S(\mathbf{x}, \mathbf{x}') u(\mathbf{x}') d\mathbf{x}' \\ &+ \int_{r < a} k^S(\mathbf{x}, \mathbf{x}') [u(\mathbf{x}') - u(\mathbf{x})] d\mathbf{x}'. \end{aligned}$$

The above definition can be extended to the case where the integration is carried out over an arbitrary surface $\mathcal{S} \in C^2$. Let $\mathbf{x} \in \mathcal{S}$ be the point at which the integrand has a second-order singularity. We consider the normal to the surface \mathcal{S} and form a circular cylinder, of radius ε , having the normal as the axis of rotation. Denote by s_ε the part of the surface enclosed in the cylinder. We take now the orthogonal projection of the surface s_ε onto the tangential plane to the surface passing through the point \mathbf{x} . Note that the mapping given by the orthogonal projection is conformal at the point \mathbf{x} . Denoting by \mathbf{x}'' the projection of the point \mathbf{x}' and by s_ε'' the disk $|\mathbf{x}'' - \mathbf{x}| \leq \varepsilon$ we can write

$$\int_{s_\varepsilon} \frac{f(\mathbf{x}, \theta)}{|\mathbf{x}' - \mathbf{x}|^2} u(\mathbf{x}') ds' = \int_{s_\varepsilon''} \frac{f(\mathbf{x}, \theta)}{|\mathbf{x}' - \mathbf{x}|^2} u(\mathbf{x}'') ds''.$$

But

$$\frac{f(\mathbf{x}, \theta)}{|\mathbf{x}' - \mathbf{x}|^2} = \frac{f(\mathbf{x}, \theta)}{|\mathbf{x}'' - \mathbf{x}|^2} + \frac{f_0(\mathbf{x}, \mathbf{x}'')}{|\mathbf{x}'' - \mathbf{x}|^\gamma}, \quad \gamma < 2.$$

Hence, by a principal value integral over the surface \mathcal{S} , we mean the expression

$$\begin{aligned} \int_{\mathcal{S}} k^S(\mathbf{x}, \mathbf{x}') u(\mathbf{x}') d\mathbf{x}' &= \int_{\mathcal{S} - s_\varepsilon} k^S(\mathbf{x}, \mathbf{x}') u(\mathbf{x}') d\mathbf{x}' \\ &+ \int_{s_\varepsilon''} \frac{f(\mathbf{x}, \theta)}{|\mathbf{x}'' - \mathbf{x}|^2} u(\mathbf{x}'') ds'' \\ &+ \int_{s_\varepsilon''} \frac{f_0(\mathbf{x}, \theta)}{|\mathbf{x}'' - \mathbf{x}|^\gamma} u(\mathbf{x}'') ds''. \end{aligned}$$

A.2. 2D singular integral equations

Consider a singular integral equation

$$\begin{aligned}
Au &= a_0(\mathbf{x}) + \int_{\Omega} k^S(\mathbf{x}, \mathbf{x}')u(\mathbf{x}') d\mathbf{x}' \\
&+ \int_{\Omega} k^R(\mathbf{x}, \mathbf{x}')u(\mathbf{x}') d\mathbf{x}' = g(\mathbf{x}),
\end{aligned}
\tag{66}$$

where $k^S(\mathbf{x}, \mathbf{x}')$ is the singular kernel defined by formula (64) and $k^R(\mathbf{x}, \mathbf{x}')$ is a regular kernel. In the case where the operator $k^S(\mathbf{x}, \mathbf{x}')$ is missing, the integral equation (66) is called regular. One of the main differences between the two types of equations is that while the usual iteration method for regular equations reduces the unbounded operators to bounded ones it fails to do so in the case of singular equations. In other words, the direct iteration method is not working for singular integral equations.

The usual approach to solving Eq. (66) consists in finding an integral operator B , of the same type as A , such that the classical Fredholm theory will apply to the equation

$$BAu = Bg. \tag{67}$$

The operator B is called a regularizing operator. The important problem is to find a regularizing operator such that Eq. (67) is equivalent to the initial one (66).

For a class of singular operators, including the operators entering in our paper, Mikhlin [13] has developed a procedure, based on a symbolic calculus of singular integrals, for determining a regularizing operator yielding an equivalent equation to (66). Let us consider the singular part of Eq. (66):

$$A^S = a_0(\mathbf{x}) + \int_{\Omega} k^S(\mathbf{x}, \mathbf{x}')u(\mathbf{x}') d\mathbf{x}'. \tag{68}$$

We expand the characteristic into a Fourier series

$$f(\mathbf{x}, \theta) = \sum_{n \neq 0} b_n(\mathbf{x}) \exp(in\theta),$$

where the b_0 term is missing due to condition (65) involved in the definition of the principal integral. The *symbol of singular operator* is the complex value function $\phi(\mathbf{x}, \theta)$ defined by relationship

$$\phi(\mathbf{x}, \lambda) = \sum_{n=-\infty}^{+\infty} a_n(\mathbf{x}) \exp(in\lambda), \quad \lambda \in \mathbb{R},$$

where

$$a_n = \frac{2\pi i^n b_n(\mathbf{x})}{n}, \quad a_{-n} = (-1)^n \frac{2\pi i^{-n} b_{-n}(\mathbf{x})}{n}, \quad n > 0.$$

It should be noted that the regular terms do not have any influence on the symbol. It is clear that the symbol of an operator is determined completely by its characteristic; conversely once the coefficients a_n are known we can determine the coefficients b_n resulting this in the operators' characteristic (up to a regular operator). The symbol corresponding to the sum of two singular operators is the sum of the corresponding operators and the symbol of the composition of two principal part singular operators is the product of their symbols.

To obtain a regularizing operator for the operator A we determine its symbol $\phi(\mathbf{x}, \theta)$. In the case

$$\phi(\mathbf{x}, \lambda) \neq 0$$

for all \mathbf{x} and λ we can define the symbol of the regularizing operator as

$$\phi_1(\mathbf{x}, \lambda) = \frac{1}{\phi(\mathbf{x}, \lambda)}.$$

The theory for singular integral equations can be extended to systems of singular equations. The characteristics of the system can be written as a matrix; this yields the symbol of the system which is the matrix formed with the symbols of each element.

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}.$$

The condition for obtaining a regularizing operator is that the determinant of the symbol matrix Φ should be different from zero for all values of \mathbf{x} and λ and a regularizing operator is Φ^{-1} . According to a theorem in [14, Theorem 5.2, p. 379] a sufficient condition for a system of singular integral equations to have an equivalent regularizing operator is that the moduli of the minors

$$\delta_1 = \phi_{11}, \quad \delta_2 = \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix}, \quad \delta_3 = \begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{vmatrix} \quad (69)$$

are to be bounded from below by a positive constant. In other words, in the case where the conditions (69) are satisfied the Fredholm theory applies to the regularized system.

All this discussion supposed that the integrating domain is carried out over a plane domain. Taking into consideration the definition of the principal part integrals for arbitrary surface $\mathcal{S} \in C^2$ it can be extended to the case where the integrals are carried out over a general surface.

Finally, we note that the fulfilling of the above conditions is unaltered by a change of variables.

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